AN ADDITIVE COMBINATORIAL TAKE ON ZETA CONSTANTS

OLOF SISASK

Many know that $\zeta(2) = \sum_{n \geqslant 1} n^{-2} = \pi^2/6$ – a remarkable identity indeed. Quite a few might also be familiar with a proof of this using infinite Fourier series. What is less known, perhaps, is that one can prove it from a discrete Fourier standpoint as well, with the added bonus of a simple combinatorial interpretation of what's going on – an interpretation that generalises. A proof might go as follows: let $A = \{0, \ldots, k-1\} \subset \mathbb{Z}_N$, N odd, and let $f = 1_A$ be the characteristic function of A. For non-zero r, we then have

$$\widehat{f}(r) = \frac{1}{N} \sum_{x=0}^{k-1} \omega^{rx} = \frac{\omega^{kr} - 1}{N(\omega^r - 1)} = \frac{\omega^{kr/2} \sin(\pi kr/N)}{N\omega^{r/2} \sin(\pi r/N)}$$

where $\omega = \mathrm{e}^{2\pi i/N}$. Also, Parseval's identity tells us that $\frac{1}{N} \sum |f(x)|^2 = \sum_{r=0}^{N-1} |\widehat{f}(r)|^2$ (as is easily verified) and so, after some simplification, we have

$$k/N = k^2/N^2 + 2 \sum_{r=1}^{(N-1)/2} \frac{\sin^2(\pi rk/N)}{N^2 \sin^2(\pi r/N)}. \qquad (*)$$

Let $\delta \in [0, 1]$ be fixed, and set $k = \lfloor \delta N \rfloor$. Define $a_r(N)$ to be 0 if $r > (N-1)/2$ and $\sin^2(\pi rk/N)/N^2 \sin^2(\pi r/N)$ otherwise, so that the sum above is just $\sum a_r(N)$. Since $2/\pi \leqslant \sin x/x \leqslant 1$ for $0 \leqslant x \leqslant \pi/2$, we see that $\sup_N a_r(N) \leqslant 1/4r^2$: we may thus use the dominated convergence theorem to deduce that

$$\lim_{N \to \infty} \sum_{r=1}^{(N-1)/2} a_r(N) = \sum_{r=1}^{\infty} \lim_{N \to \infty} a_r(N) = \sum_{r=1}^{\infty} \frac{\sin^2(\pi \delta r)}{\pi^2 r^2}.$$

It therefore follows that, upon letting $N \to \infty$ in $(*)$,

$$\sum_{r=1}^{\infty} \frac{\sin^2(\pi \delta r)}{r^2} = \pi^2 \delta(1-\delta)/2.$$

Putting $\delta = 1/2$ allows us to recover our original result.

So where did the combinatorics come in? Well, there is actually a hidden count going on above, in Parseval's identity. The number of solutions to $c_1 a_1 + \cdots + c_m a_m = 0$ with $a_i \in A \subset \mathbb{Z}_N$ is easily seen to be $N^{m-1} \sum_{r=0}^{N-1} \widehat{f}(c_1 r) \cdots \widehat{f}(c_m r)$; Parseval simply says that $N \sum |\widehat{f}(r)|^2$ is the number of solutions to $a_1 = a_2$ with $a_i \in A$, also known as $|A|$. The number of solutions to $a_1 + a_2 = a_3 + a_4$ in $\{0, \ldots, k-1\} \subset \mathbb{Z}_N$ is easily computed: it is approximately $2k^3/3$ for $k < N/2$. Running through just as above, this observation yields

$$\sum_{r=1}^{\infty} \frac{\sin^4(\pi \delta r)}{r^4} = \pi^4 \delta^3 (2/3 - \delta)/2 \quad \text{for } 0 \leqslant \delta \leqslant 1/2,$$

from which we deduce $\zeta(4) = \sum_{n \geqslant 1} n^{-4} = \pi^4/90$ as a special case.

The interested reader may wish to try to obtain an expression involving the reciprocals of cubes; try counting three-term progressions, i.e., solutions to $a_1 + a_3 = 2a_2$.

Department of Mathematics, University of Bristol, Bristol BS8 1TW, England
E-mail address: `O.Sisask@dpmms.cam.ac.uk`